PROGRESSIONS

SEQUENCE

A succession of numbers $a_1, a_2, a_3, ..., a_n$ formed according to some definite rule is called a sequence.

ARITHMETIC PROGRESSION (A.P.)

A sequence of number $\{a_n\}$ is called an arithmetical progression, if there is a number d, such that $d = a_n - a_{n-1}$ for all n; and d is called as common difference (c.d).

Useful Formulae

If a = first term, d = common difference and n is the number of terms, then

(a) nth term is denoted by t_n and is given by

$$t_n = a + (n - 1)d.$$

(b) Sum of first n terms is denoted by S_n and is given by

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

or $S_n = \frac{n}{2}(a + l)$, where l = last term in the series i.e. $l = t_n = a + (n - 1)d$.

(c) Arithmetic mean A of any two numbers a and b is given by

$$A = \frac{a+b}{2}$$

Also $A = \frac{1}{n}(a_1 + a_2 + ... + a_n)$ is arithmetic mean of n numbers $a_1, a_2, ..., a_n$

(d) Sum of first n natural numbers ($\sum n$)

$$\Sigma n = \frac{n(n+1)}{2} \text{ where, } n \in N.$$

(e) Sum of first n odd natural numbers $\Sigma(2n-1)$

$$\Sigma(2n-1) = 1+3+5+...+(2n-1) = n^2$$

(f) Sum of first n even natural numbers $(\Sigma 2n)$

$$\Sigma 2n = 2+4+6+.....+2n = n(n+1)$$

(g) Sum of squares of first n natural numbers ($\sum n^2$)

$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

(h) Sum of cubes of first n natural numbers ($\sum n^3$)

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(i) Sum of fourth powers of first n natural numbers (Σn^4)

$$\Sigma n^4 = 1^4 + 2^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

(j) If terms are given in A.P., and their sum and product are known, then the terms must be picked up in following way in solving certain problem.

- For three terms (a – d), a, (a + d)
- For four terms (a – 3d), (a – d), (a + d), (a + 3d)
- For five terms (a – 2d), (a – d), a (a + d), (a + 2d)

USEFUL PROPERTIES

(a) If $t_n = an + b$, then the series so formed is an A.P.

(b) If $S_n = an^2 + bn + c$, then series so formed is an A.P.

(c) If every term of an A.P. is increased or decreased by the same quantity, the resulting terms will also be in A.P.

(d) If every term of an A.P. is multiplied or divided by the same non-zero quantity, the resulting terms will also be in A.P.

(e) If terms $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n+1}$ are in A.P. Then sum of these terms will be equal to $(2n + 1)a_{n+1}$. Here total number of terms in the series is (2n + 1) and middle term is a_{n+1}.

(f) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.

(g) Sum and difference of corresponding terms of two A.P.'s will form a series in A.P.

(h) If terms $a_1, a_2, \ldots, a_{2n-1}, a_{2n}$ are in A.P. The sum of these terms will be equal to $(2n)\left(\frac{a_n + a_{n+1}}{2}\right)$,

where $\frac{a_n + a_{n+1}}{2}$ = A.M. of middle terms.

(i) nth term of a series is $a_n = S_n - S_{n-1}$ $(n \geq 2)$

GEOMETRIC PROGRESSION (G.P.)

The sequence $\{a_n\}$ in which $a_1 \neq 0$ is termed a geometric progression if there is a number $r \neq 0$ such that $\frac{a_n}{a_{n-1}} = r$ for all n, then r is called common ratio.

Useful Formulae

If a = first term, r = common ratio and n is the number of term, then

(a) n^{th} term denoted by t_n is given by

$t_n = ar^{n-1}$

(b) Sum of first n terms denoted by S_n is given by

$S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$ corresponding to r < 1 (or) r > 1, (or) $S_n = \frac{a - rl}{1-r}$

where l is the last term in the series.

(c) Sum of infinite terms (S_∞)

$$S_n = \frac{a}{1-r} \quad (\text{For } |r| < 1)$$

(d) Geometric mean (G)

(i) $G = \sqrt{ab}$ where a, b are two positive numbers.

(ii) $G = (a_1 a_2 \ldots\ldots a_n)^{1/n}$ is geometric mean of n positive numbers $a_1, a_2, a_3, \ldots\ldots a_n$.

(e) If terms are given in G.P. and their product is known, then the terms must be picked up in following way.

- For three terms $\frac{a}{r}$, a, ar
- For four terms $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3
- For five terms $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2

USEFUL PROPERTIES

(a) The product of the terms equidistant from the beginning and end is constant. And it is equal to the product of first and last terms.

(b) If every term of G.P. is increased or decreased by the same non–zero quantity, the resulting series may not be in G.P.

(c) If every term of G.P. is multiplied or divided by the same non–zero quantity, the resulting series is in G.P.

(d) If a_1, a_2, a_3 ..... and b_1, b_2, b_3 ..... be two G.P.'s of common ratio r_1 and r_2 respectively, then

a_1b_1, a_2b_2,....... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ .... will also form G.P. common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

(e) If a_1, a_2, a_3, ... be a G.P. of positive terms, then $\log a_1$, $\log a_2$, $\log a_3$, ... will be in A.P. and conversely.

Let $b = ar, c = ar^2$ and $d = ar^3$. Then, a, b, c, d are in G.P.

HARMONIC PROGRESSION (H.P.)

A sequence is said to be a harmonic progression, if and only if the reciprocal of its terms form an arithmetic progression.

SOME USEFUL FORMULAE & PROPERTIES

(a) n^{th} term of H.P. = $\frac{1}{n^{th}\text{ term of AP}}$

(b) Harmonic mean H of any two numbers a and b is given by

$H = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b}$ where a, b are two non–zero numbers.

Also $H = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}} = \frac{n}{\sum_{j=1}^{n}\frac{1}{a_j}}$

for the harmonic mean of n non–zero numbers a_1, a_2, a_3,....., a_n.

(c) If terms are given in H.P. then the terms could be picked up in the following way

- For three terms

$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

- For four terms

$\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

- For five terms

$$\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$$

INSERTION OF MEANS BETWEEN TWO NUMBERS

If a and b are two given numbers.

ARITHMETIC MEANS

Let a, A_1, A_2, ... A_n, b be in A.P. then A_1, A_2, ... A_n are n A.M. 's between a and b. If d is common difference, then

$$b = a + (n + 2 - 1)\, d \Rightarrow d = \frac{b-a}{n+1}$$

$$A_1 = a + d = a + \frac{b-a}{n+1} = \frac{an+b}{n+1}$$

$$A_2 = a + 2d = a + 2\,\frac{b-a}{n+1} = \frac{a(n-1)+2b}{n+1}$$

$$A_3 = a + 3d = a + 3\,\frac{b-a}{n+1} = \frac{a(n-2)+3b}{n+1}$$

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$$A_n = a + nd = a + n\frac{(b-a)}{n+1} = \frac{a+nb}{n+1}$$

Note : The sum of n A.M's, $A_1 + A_2 + ... + A_n = \frac{n}{2}(a + b)$.

GEOMETRIC MEANS

Let a, G_1, G_2...... G_n, b be in G.P., then G_1, G_2 ....G_n are n G.M.s between a and b. If r is a common ratio, then

$$b = a\, r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$$

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}} = a^{\frac{n}{n+1}}\, b^{\frac{1}{n+1}}$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}} = a^{\frac{n-1}{n+1}}\, b^{\frac{2}{n+1}}$$

$$G_3 = ar_3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}} = a^{\frac{n-2}{n+1}}\, b^{\frac{3}{n+1}}$$

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$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} = a^{\frac{1}{n+1}}\, b^{\frac{n}{n+1}}$$

Note : The product of n G.M's $G_1\, G_2$ .... $G_n = \left(\sqrt{ab}\right)^n$

HARMONIC MEANS

If a, H_1, H_2 .... H_n, are in H.P., then H_1, H_2...........H_n are the n H.M.'s between a and b. If d is the common difference of the corresponding A.P. then

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)\ d \Rightarrow d = \frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)} = \frac{bn+a}{ab(n+1)}$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + \frac{2(a-b)}{ab(n+1)} = \frac{b(n-1)+2a}{ab(n+1)}$$

$$\frac{1}{H_3} = \frac{1}{a} + 3d = \frac{1}{a} + \frac{3(a-b)}{ab(n+1)} = \frac{b(n-2)+3a}{ab(n+1)}$$

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$$\frac{1}{H_n} = \frac{1}{a} + nd = \frac{1}{a} + \frac{n(a-b)}{ab(n+1)} = \frac{b(1)+na}{ab(n+1)}$$

RELATION BETWEEN A, G AND H

If A, G and H are A. M., G.M. and H.M. of two positive numbers a and b, then

(i) $G^2 = AH$, (ii) $A \geq G \geq H$

Note :

(1) For given n positive numbers a_1, a_2, a_3,.......... a_n, A.M. $\geq$ G.M. $\geq$ H.M. . The equality holds when the numbers are equal.

(2) If sum of the given n positive numbers is constant then their product will be maximum if numbers are equal.

ARITHMETICO–GEOMETRIC SERIES

The series whose each term is formed by multiplying corresponding terms of an A.P. and G.P. is called the Arithmetico–geometric series.

For Examples

- $1 + 2x + 4x^2 + 6x^3 + \ldots\ldots$
- $a + (a + d)r + (a + 2d)r^2 + \ldots\ldots$

SUMMATION OF n TERMS OF ARITHMETICO–GEOMETRIC SERIES

Let $S = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots$

(i) $t_n = [a + (n - 1)d].r^{n-1}$

(ii) $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots + [a + (n - 1)d]r^{n-1}$

Multiply by 'r' and rewrite the series in following way.

$rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + \ldots\ldots + [a + (n - 2)d]r^{n-1} + [a + (n - 1)d]r^n$

On subtraction,

$S_n(1 - r) = a + d\,(r + r^2 + \ldots\ldots + r^{n-1}) - [a + (n - 1)d]r^n$

or, $S_n(1-r) = a + \dfrac{dr(1-r^{n-1})}{1-r} - [a + (n-1)d].r^n$

or, $\boxed{S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]}{1-r}.r^n}$

SUMMATION OF INFINITE TERMS SERIES :

$S = a + (a + d)r + (a + 2d)r^2 + ...\ \infty$

$rS = \qquad a\,r + (a + d)\,r^2 + ...\text{to}\ \infty$

On subtraction

$S\,(1 - r) = a + d\,(r + r^2 + r^3 + ...\infty)$

$$S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

DIFFERENCE METHOD

Let T_1, T_2, T_3 ...... T_n are the terms of sequence, then

(i) If $(T_2 - T_1), (T_3 - T_2)$............... $(T_n - T_{n-1})$ are in A.P. then, the sum of the such series may be obtained by using summation formulae in nth term,

(ii) If $(T_2 - T_1), (T_3 - T_2)$ .... $(T_n - T_{n-1})$ are found in G.P. then the sum of the such series may be obtained by using summation formulae of a G.P.